Lecture 18

Equivalence Relation and Partial Ordering

Equivalence Relation

Definition: A relation R on a set A is called **equivalence relation** if it is reflexive, symmetric, and transitive.

Example: Let m be an integer $m > 1$. Is R an equivalence relation on $\mathbb{Z}$?

$$R = \{(a, b) \mid a \equiv b \pmod{m}\}$$

Solution: We know that $a \equiv b \pmod{m}$ if and only if $m \mid a - b$.

R **is reflexive:** $(a, a) \in R$ because $m \mid a - a$.

R **is symmetric:** $(a, b) \in R \implies m \mid (a - b) \implies m \mid (b - a) \implies (b, a) \in R$

R **is transitive:** $(a, b) \in R$ and $(b, c) \in R \implies m \mid (a - b)$ and $m \mid (b - c)\ldots$

$$\ldots \implies m \mid a - c \implies (a, c) \in R$$

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Equivalence Relation

Example: Is R on $\mathbb{Z}$ an equivalence relation?

$$R = \{(a, b) \mid a \text{ divides } b\}.$$

Solution: No, because $a \mid b$ does not imply $b \mid a$. Hence, R is not symmetric. ∎

Example: Is R on $\mathbb{R}$ an equivalence relation?

$$R = \{(a, b) \mid |a - b| < 1\}.$$

Solution: No, because $(.2,.9), (.9,1.8) \in R$ but $(.2,1.8) \notin R$. Hence, R is not transitive. ∎

Equivalence Class

Definition: Let R be an equivalence relation on a set A. For any $x \in A$, the set of all elements of A that are related to x is called the **equivalence class** of x and denoted by $[x]$.

Example: Let $R = \{(a, b) \mid a \equiv b \pmod 4\}$ be an equivalence relation on $\mathbb{Z}$.
Find $[0]$, $[2]$, and $[6]$.

Solution:

$$[0] = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$$

$$[2] = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$$

$$[6] = \{\ldots, -2, 6, 6, 10, 14, \ldots\}$$

Observe that $[2] = [6]$ and $[2] \cap [0] = \varnothing$

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Equivalence Class

Theorem: Let R be an equivalence relation on a set A. These statement for element a and b of A are equivalent: (i) aRb (ii) $[a] = [b]$ (iii) $[a] \cap [b] \neq \emptyset$

Proof Sketch: It's enough to show that $(i) \implies (ii) \implies (iii) \implies (i)$.

Show $(i) \implies (ii)$ using transitivity and symmetry.

Show $(ii) \implies (iii)$ using reflexivity.

Show $(iii) \implies (i)$ using transitivity and symmetry.

∎

Partition

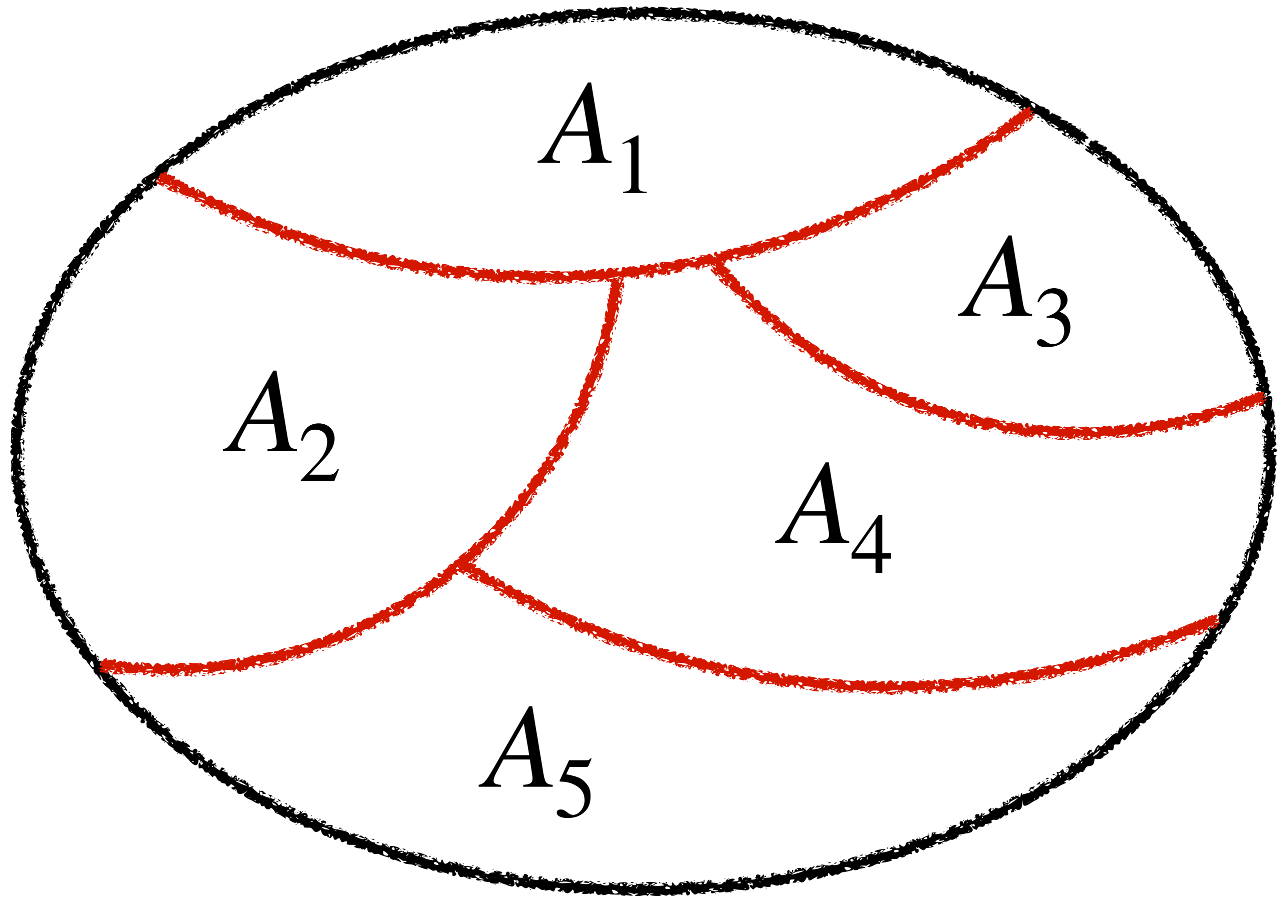
Definition: A collection of subsets $A_1, A_2, \ldots, \subseteq S$ forms a partition of S if:

(i) $A_i \neq \emptyset$

(ii) $A_i \cap A_j = \emptyset$, if $i \neq j$,

(iii) $\cup_i A_i = S$.

Partitions of S



Example: Set of even integers and set of odd integers form a partition of $\mathbb{Z}$.

Equivalence Class and Partition

Theorem: If we have an equivalence relation R on A, the equivalence classes of relation form a partition of A.

Proof: Let $A_1, A_2, A_3, \ldots$ are the distinct equivalence classes created by R on A

$A_i \neq \emptyset$:

Each A_i is defined with respect to an element a_i and $a_i \in A_i$ because $(a_i, a_i) \in R$.

$A_i \cap A_j = \emptyset$, if $i \neq j$:

WLOG let $a_i \in A_i$, $a_j \in A_j$ such that $a_i \notin A_j$. Then, $(a_i, a_j) \notin R$. Hence, $A_i \cap A_j = \emptyset$.

$\cup_i A_i = A$:

Every element $a \in A$ belongs to some A_i. Therefore, $\cup_i A_i = A$. ∎

Partial Ordering

Definition: A relation R on a set S is called **partial ordering** or **partial order** if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a **partially ordered set** or **poset**, and is denoted by (S, R).

Example: Is $R = \{(a, b) \mid a \text{ divides } b\}$ on $\mathbb{Z}^+$ a partial ordering? **Yes.**

Example: Is $R = \{(a, b) \mid a \text{ is older than } b\}$ on set of people a partial ordering? **No.**

Notation: $a \preceq b$ denotes $(a, b) \in R$ in a poset (S, R).

$a \prec b$ denotes $(a, b) \in R$, but $a \neq b$ in a poset (S, R).

Total Ordering

Definition: Elements a and b of a poset (S, R) are called **comparable** if either $a \preceq b$ or $b \preceq a$.

Note: Not all elements of a poset are comparable, hence the name "partial" ordering.

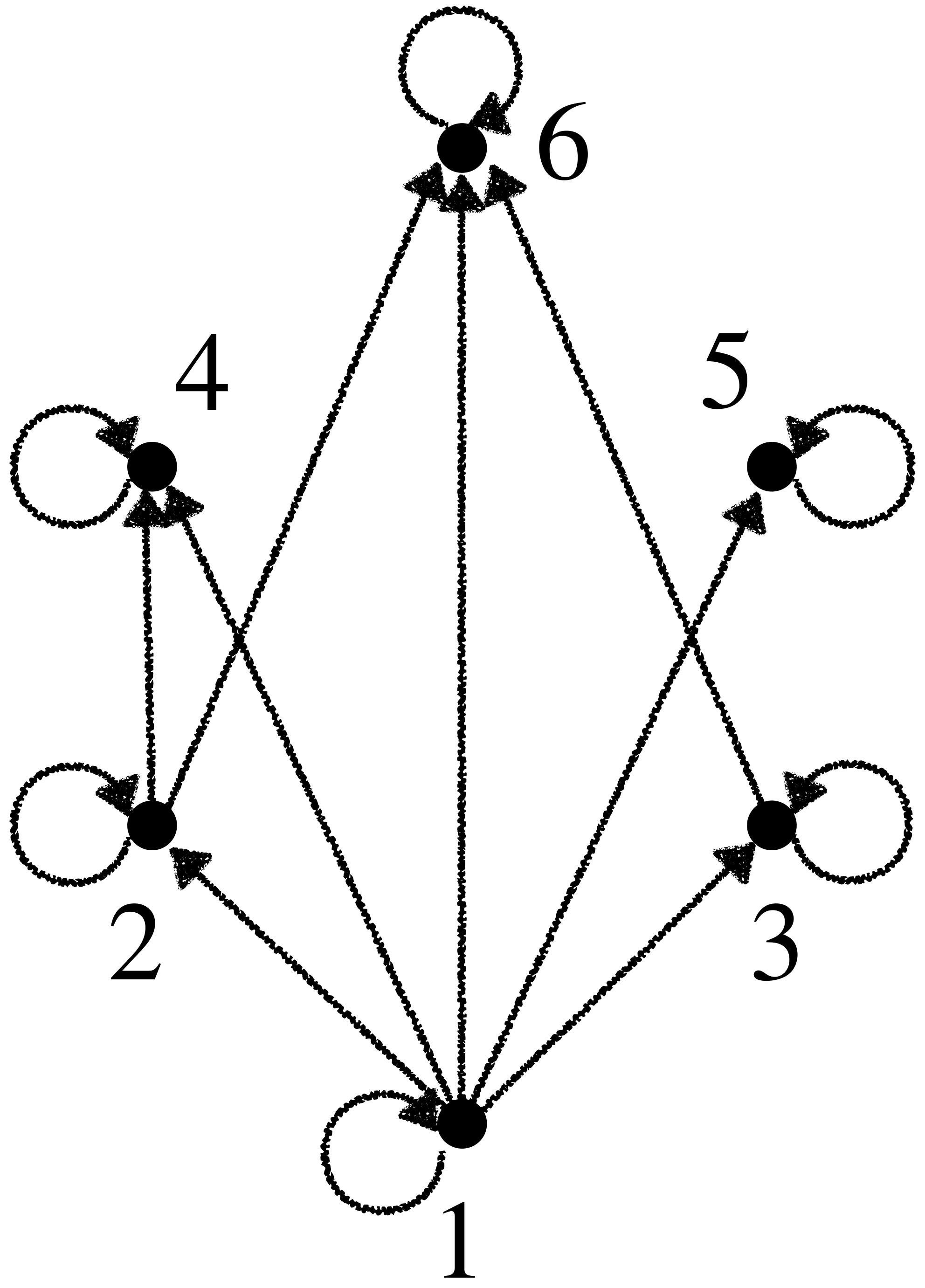
Definition: If (S, R) is a poset, where every two elements are comparable, S is called a **totally ordered set**.

Example: Is $R = \{(a, b) \mid a \leq b\}$ on $\mathbb{Z}$ a total ordering? **Yes.**

Hasse Diagram

Consider poset (X, R), where $X = \{1,2,3,4,5,6\}$, and $R = \{(a, b) \mid a \text{ divides } b\}$.

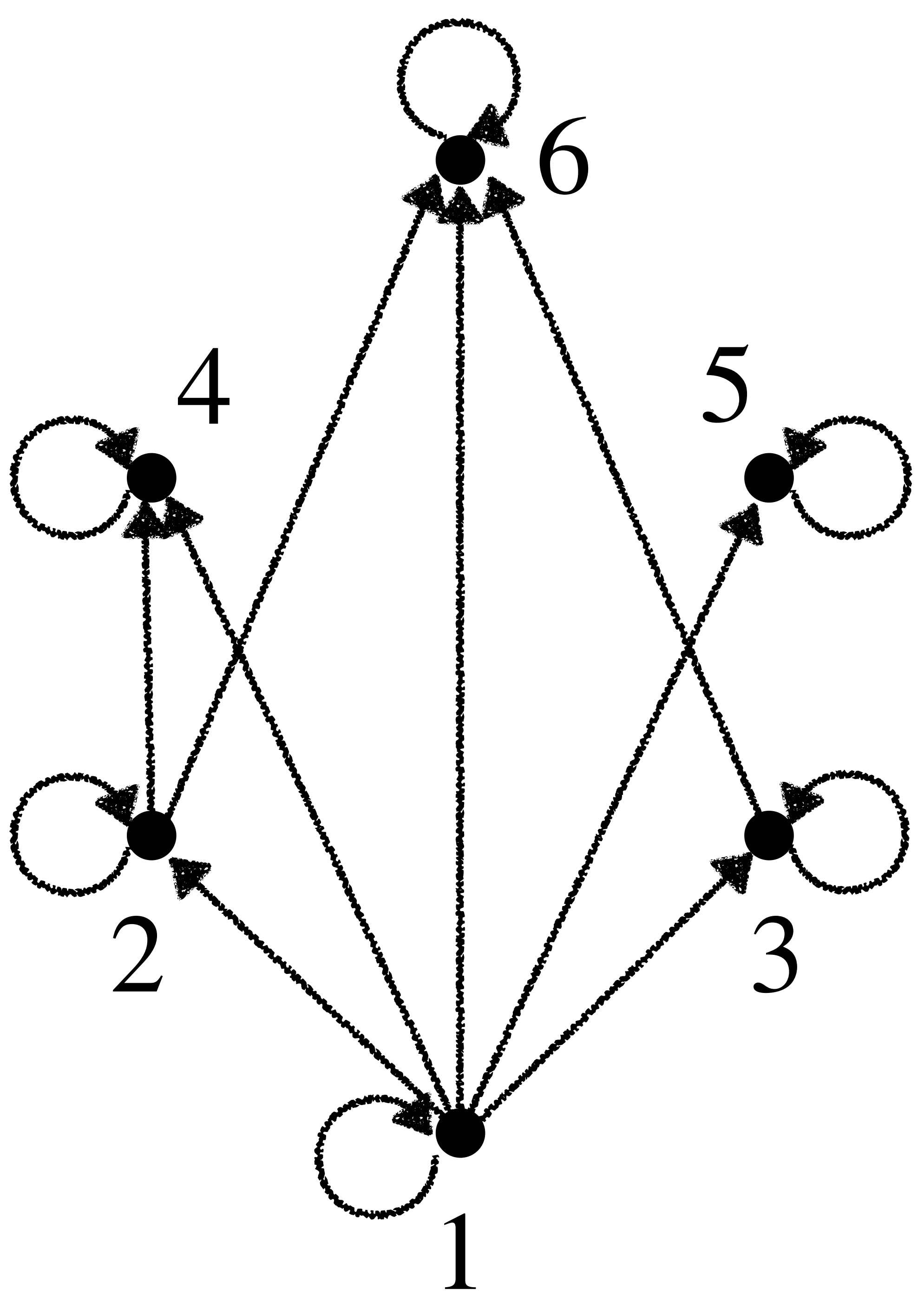
Step 1: Draw ● for every element of X, and put an arrow from a to b, if $(a, b) \in R$.



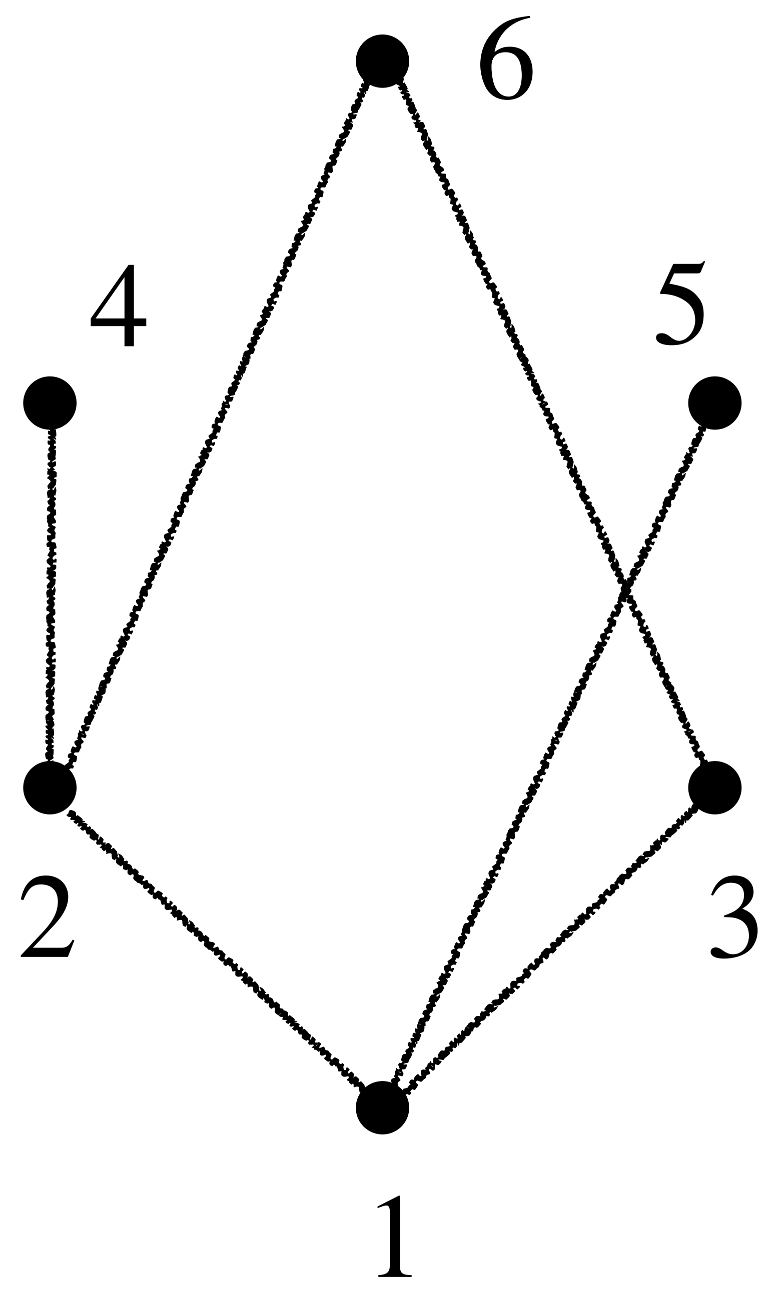
Hasse Diagram

Why we can ensure this?

Step 2: Remove self-loops and arrows that must be present because of transitivity.



Step 3: Arrange arrows such that they are pointing upwards and remove direction.



Special Elements of a Partially Order

Definition: An element a is **maximal** in the poset $(S, \preceq)$ if there is no $b \in S$ such that $a \prec b$.

Definition: An element a is **minimal** in the poset $(S, \preceq)$ if there is no $b \in S$ such that $b \prec a$.

Note: Maximal and minimal may not exist and they need not be unique when they exist.

Definition: An element a is **greatest** in the poset $(S, \preceq)$ if $b \preceq a, \forall b \in S$.

Definition: An element a is **lowest** in the poset $(S, \preceq)$ if $a \preceq b, \forall b \in S$.

Note: Greatest and lowest may not exist, but when they exist they are unique.